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CONTENTS.

A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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we have the matrix representation

$$H(X) = | \bar{X}_1 \bar{X}_2 \dots \bar{X}_n | . | a_{ij} | . \begin{vmatrix} X_1 \\ X_2 \\ \dots \\ X_n \end{vmatrix}$$

If r is the rank of $| a_{ij} |$, it is known from the theory of Hermitian matrices, that there exists a complex matrix S, such that

$$\bar{S}' . | a_{ij} | . S = [\pm 1]_r.$$

Hence $H(X)$ can be reduced to the form

$$\varepsilon_1 Y_1 \bar{Y}_1 + \varepsilon_2 Y_2 \bar{Y}_2 + \dots + \varepsilon_r Y_r \bar{Y}_r, \qquad \dots \quad (1)$$

where Y's are independent linear forms in the X's, and ε's are ± 1. We take as before the signature s, namely the number of negative ε's, to be equal to or less than $\frac{1}{2}r$.

A ***solution*** of $H(X)$ is a set of complex values of $x_1, x_2, \dots x_n$ which make $H(X)$ vanish. An ***imbedded solution of order*** k is a set of k linearly independent solutions of $H(X)$, such that every linear combination of them (with complex co-efficients) is also a solution. We shall now interpret the signature of $H(X)$, by shewing that $n - r + s$ is the maximum order of imbedded solutions possessed by $H(X)$.

If A, B are any two complex numbers, we have the identity,

$$(A - B)(\bar{A} + \bar{B}) + (\bar{A} - \bar{B})(A + B) = 2(A\bar{A} - B\bar{B}) \dots \quad (2)$$

Hence if there are $s \ (\leqslant \frac{1}{2}r)$ negative ε's in (1), we can by using the identity (2) transform it into ... (3)

$$X_1\bar{X}_1 + X_2\bar{X}_2 + \dots + X_{r-2s}\bar{X}_{r-2s} + (Y_1\bar{Z}_1 + \bar{Y}_1 Z_1) + \dots + (Y_s\bar{Z}_s + \bar{Y}_s Z_s),$$

where $X_1, \dots, X_{r-2s}, Y_1, \dots, Y_s, Z_1, \dots, Z_s$ are r independent linear forms in the original variables. It is clear from this form that $H(X)$ possesses imbedded solutions of order $n - r + s$, of the form

$$X_1 = X_2 = \dots = X_{r-2s} = Y_1 = \dots = Y_s = 0.$$

Conversely if H (X) is known to possess an imbedded solution of order $n-t$, we could shew by a procedure similar to that adopted for the quadratic form, that $s \geqslant r-t$, thus establishing that $n-r+s$ is the maximum order of imbedded solutions of H (X).

IV. For the parallel theory of double-polynomials, we have to consider the Hermitian polynomial :

$$H(z, \bar{z}) = \Sigma\, a_{ij} z^{i-1} \bar{z}^{j-1}; \qquad (i, j = 1, 2, \ldots, n;\ a_{ij} = \bar{a}_{ji}).$$

This equated to zero determines a *cyclic curve of order* $n-1$, in the plane of the complex variable z. By a parallel procedure, if $H(z, \bar{z})$ is of rank r, it could be reduced to the form

$$\varepsilon_1 f_1(z) \bar{f}_1(\bar{z}) + \ldots + \varepsilon_r f_r(z) \bar{f}_r(\bar{z}),$$

where the ε's are ± 1, and f's are linearly independent polynomials. If $s\ (\leqslant \frac{1}{2}r)$ of the ε's are negative, then on using the identity of the last paragraph, we have the canonical form for $H(\bar{z}, z)$:

$$H(z, \bar{z}) = \sum_{k=1}^{r-2s} f_k(z) \bar{f}_k(\bar{z}) + \sum_{k=1}^{s} \{ \phi_k(z) \bar{\psi}_k(\bar{z}) + \bar{\phi}_k(\bar{z}) \psi_k(z) \}$$

where the f's, ϕ's and ψ's form a set of r linearly independent polynomials.

A complex polynomial $\phi(z)$ of order $n-1$, equated to zero, determines a group of $n-1$ points in the complex plane ; these points considered as a degenerate case of a cyclic of order $n-1$, are obtained by equating to zero the Hermitian form $\phi(z)\, \bar{\phi}(\bar{z})$. The condition that the group of points is apolar to the cyclic determined by $H(z, \bar{z})$ will therefore be the vanishing of a Hermitian bilinear form in the coefficients of ϕ, which as before, we may easily show to possess the same rank r and the same signature s as $H(z, \bar{z})$. Since we have shown that the maximum order of imbedded solutions possessed by a Hermitian bilinear form is $n-r+s$, it follows that the number of linearly independent apolar groups of $(n-1)$ points, possessed by the cyclic $H(z, \bar{z}) = 0$ of order $n-1$, is equal to $n-r+s$.

We may in this connection mention also that a Hermitian polynomial of rank r can be brought uniquely to the shape

$$\sum_{k=1}^{r} \varepsilon_k f_k(z) \bar{f}_k(\bar{z}),$$

where each ε is ± 1 and f_p, f_q are apolar for $p \neq q$. The proof of this is easy, and depends on the fact that any two Hermitian forms can be brought simultaneously to the 'diagonal' form.

As a particular case it would follow that a real symmetric double-polynomial can be brought to the shape

$$\Sigma \ \varepsilon_k f_k(x) f_k(y),$$

where every two of the real polynomials f are apolar.

It follows that the apolar linear systems of maximum order, of a polynomial $F(x, y)$ fall into two distinct systems in the two cases:

(1) when $F(x, y)$ is a symmetric complex double-polynomial of even rank;

(2) when $F(x, y)$ is a symmetric real double polynomial of signature r and rank $2r$.

Even though Hermitian bilinear forms are a generalisation of real quadratic forms, still the imbedded solutions of maximum order of a Hermitian bilinear form of rank $2r$ and signature r do *not* fall into two algebraically distinct systems. We may see how this is so by considering the simple Hermitian form

$$X_1 \bar{X}_1 - X_2 \bar{X}_2.$$

The solutions of this (which are also imbedded solutions of maximum order) are

$$X_1 = xe^{i\theta_1}, \quad X_2 = xe^{i\theta_2},$$

where x is real and positive. These solutions, do not evidently fall into two distinct systems. But if we require the solutions to be real (in which case they would be solutions of the real quadratic form $X_1^2 - X_2^2$,

then θ_1, θ_2 must be equal to 0 or π, and we have two algebraically distinct systems, according as $\theta_1 = \theta_2$ or $\theta_1 \neq \theta_2$.

It follows that the apolar linear systems of a Hermitian polynomial do not fall into distinct systems.

V. *Cases of two algebraically distinct systems of imbedded solutions of maximum order.*

In the case of complex quadratic forms in n variables, of rank r, the maximum order of imbedded solutions is always $n - r + [r/2]$ (where $[x]$ denotes the greatest integer in x) and these fall into two algebraically distinct systems if r is even.

In the case of real quadratic forms in n variables, of rank $2r$ and signature r, it is known that the real imbedded solutions of maximum order $n - r$, fall into two algebraically distinct systems characterised by certain intersection relations.

Though the Hermitian form is a generalisation of the real quadratic form, it is interesting to observe that this bifurcation under similar circumstances does not occur in the case of the Hermitian form. To illustrate how this happens, consider the Hermitian bilinear form

$$X\bar{Y} + \bar{X}Y,$$

which is of rank 2 and signature 1. The imbedded solutions of order $n - 1$ are given by

$$X + i\lambda Y = 0,$$

where λ is real. These obviously do not fall into two distinct systems. If however X, Y are real, the Hermitian form becomes a real quadratic form of the same rank and signature. Among the imbedded solutions, only two are real, namely $X = 0$ and $Y = 0$; these correspond to the two algebraically distinct systems of imbedded solutions of a real quadratic form of rank-signature $(2r, r)$.

VI. *A canonical shape for Hermitian forms.*

In the paper 'On the Rank of the Double-binary Form' (loc. cit.), it was shown that a double-binary form $F(x, y)$ of rank r, both

whose partial orders are even, can under certain conditions be reduced ***uniquely*** to the shape

$$f_1(x)\ \phi_1(y) + \dots + f_r(x)\ \phi_r(y),$$

where every two f's and every two ϕ's are mutually apolar. It will now be shewn that a similar reduction is valid ***unconditionally*** for the Hermitian double-polynomial of degree $2n$.

Let H $(z, \bar{z})$ of degree $2n$ in z and of rank r be reduced to the form

$$\varepsilon_1 f_1(z) \bar{f_1}(\bar{z}) + \dots + \varepsilon_r f_r(z) \bar{f_r}(\bar{z}),$$

where each ε is ± 1. Write

$$f_p(z) = \sum_q a_{pq}\ \phi_q(z) \qquad (p, q, = 1, 2, \dots r).$$

Then if H is to be unaltered in form, we must have

$$|\bar{a}'| \,.\, |\varepsilon| \,.\, |a| = |\varepsilon|, \qquad \dots \quad (1)$$

where $|\varepsilon|$ is the matrix all of whose elements vanish, except the diagonal elements, which are $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r$, $|a|$ is the matrix $|a_{pq}|$ and $|\bar{a}'|$ is the complex conjugate of the transposed matrix of $|a|$.

Now let

$$d_{pq} = (f_p, \bar{f}_q)^{2n}, \qquad D_{pq} = (\phi_p, \bar{\phi}_q)^{2n}.$$

Then

$$d_{pq} = |a| \,.\, |D_{pq}| \,.\, |\bar{a}'|,$$

so that

$$|\bar{a}'| \,.\, |\varepsilon| \,.\, |a| = |\varepsilon|$$

$$|\bar{a}'| \,.\, |d_{pq}|^{-1} \,.\, |a| = |D_{pq}|^{-q} \qquad \dots \quad (2)$$

Now any two Hermitian forms have only linear invariant factors and can therefore be simultaneously reduced to the canonical form. Since the f's are of even degree $2n$, the matrix $|d_{pq}|$ is a Hermitian matrix. Thus the two matrices $|\varepsilon|$ and $|d_{pq}|^{-1}$ can be simultaneously reduced to the diagonal form by a single Hermitian transformation. Thus there is one and only one way of choosing $|a|$, so that the canonical shape of H $(z, \bar{z})$ is preserved, and at the same time the polynomials ϕ satisfy the relation $(\phi_p, \bar{\phi}_q)^{2n} = 0$ for $p \neq q$. For example, the equation to a

cyclic of the second order, can be reduced uniquely to the form :

$$f_1(z)\bar{f_1}(\bar{z}) + f_2(z)\bar{f_2}(\bar{z}) \pm f_3(z)\bar{f_3}(\bar{z}) = 0,$$

where the quadratics f_1 f_2, f_3 are so related that f_1 is apolar to $\bar{f_2}$ and f_3 and f_2 apolar to $\bar{f_3}$.

It would also follow by similar procedure, that a real symmetric double-polynomial F (x, y), of rank r, signature s, and even degree $2n$ in x (or y,) can be reduced uniquely to the shape:

$$f_1(x)f_1(y) + \dots + f_{r-s}(x)f_{r-s}(y) \\ - f_{r-s+1}(x)f_{r-s+1}(y) - \dots - f_r(x)f_r(y),$$

where the f's are real polynomials every two of which are mutually apolar.

For example, a real symmetric double-quadratic in x, y, of rank **3**, can be reduced uniquely to the shape

$$f_1(x)f_1(y) + f_2(x)f_2(y) \pm f_3(x)f_3(y),$$

where f_1, f_2, f_3 are three real quadratics every two of which are apolar,

GENERALISATION OF THE POTENTIALS OF A SURFACE OF DOUBLE DISTRIBUTION IN EUCLIDEAN SPACE.

By K. Nagabhushanam.

1. It is well-known that the field due to a magnetic shell at a point outside it is capable of a two-fold representation, *viz.*, (1) as the negative of the gradient of a scalar potential ψ and (2) as the curl of a vector potential (P_i).

The object of the present paper is to examine this two-fold representation of the field due to a p-dimensional surface of double distribution lying in Euclidean Space of n-dimensions ($p < n$), at a point outside it, and to inquire into the conditions necessary and sufficient for such a representation. In the preparation of this paper I am greatly indebted to Dr. R. Vaidyanathaswamy for his kind guidance and valuable suggestions.

2. Let A and B represent two equal and opposite charges $+\rho$ and $-\rho$ lying very close together. Let O be a point whose distances from A and B are r_1 and r_2 respectively. Let the law of force between two unit similar charges distant r apart be $f(r)$. Let $\phi(r) = \int f(r)\,dr$. The scalar potential due to the elementary magnet ($+\rho$ at A and $-\rho$ at B) at O is the work done in bringing a unit positive charge from a standard position to O, and is denoted by

$$\psi = \rho\left\{\phi(r_2) - \phi(r_1)\right\} = \rho\int_B^A \text{grad}\ \phi(r)\,.\,d\mathbf{r}$$

where $d\mathbf{r}$ is the directed line-element and the dot after grad $\phi(r)$ denotes the inner product.

The last expression $= \rho$ inner product of grad $\phi(r)$ and the vector BA (since A and B are very near).

Hence the scalar potential may be regarded as the work done in taking a positive charge ρ from B to A while the unit charge at O is kept fixed.

The directed line-element BA may be called the displacement.

3. We shall next define a magnetic shell or a surface of double distribution* in three dimensions and obtain the scalar potential due to it at a point O outside it. Let a surface σ be given with a continuously turning tangent plane and with a continuous function μ of position of a point Q on it. Any element of the surface whose extent is $\Delta\,\sigma$ may be supposed to correspond to a magnet having its axis along the normal and having a moment $= \mu\,.\,\Delta\,\sigma$. Such a surface is called a magnetic shell. The physical interpretation is as follows. Consider two parallel surfaces of surface densities λ and $-\lambda$ at distance d apart: let them be made to approach nearer and nearer such that λd is always equal to μ, a finite quantity at each point of the surface. Then the potential and the field due to the combined effect of these two surfaces tend to definite limits and are called the potential and the field of the double layer.

$$\text{The scalar-potential} = \int \lambda d\,\Delta\sigma\,.\,\mathbf{n}\,.\,\text{grad}\;\phi\;(r)$$
$$= \int \mu\,\Delta\sigma\,.\,\mathbf{n}\,.\,\text{grad}\;\phi\;(r)$$

where $\mathbf{n}$ denotes the unit normal vector and the dot after it, the scalar product.

The expression $\mu\;\Delta\sigma\;\mathbf{n}$ denotes the value of the charge multiplied by the normal element which indicates the direction of the displacement by which charge may be taken to be moved from one side to the other of the shell. Grad $\phi\;(r)$ will be called the force vector.

4. Lower Tensor Potential.—

The foregoing ideas are directly generalisable to the case of a continuous p-dimensional surface σ associated with a continuous function μ of position of a variable point Q on it, immersed in a Euclidean Space of n dimensions with a given metrical ground-form

$$ds^2 = g_{ik}\;dx^i\;dx^k\;\dagger$$

The determinant of the co-efficients of this form will be denoted by g. It is a scalar of weight two. Now consider an element of the surface

* For a discussion of a surface of double distribution reference may be made to O. D. Kellog: *Foundations of Potential Theory*, p. 66.

† The criterion for space to be Euclidean is that ds^2 can be reduced to the sum of n squares.

at Q $(x^1\ x^2\ \dots\ x^n)$ whose direction is given by the p infinitesimal vectors $d_1, d_2 \dots d_p$. It may be represented by

$$d\sigma^{\alpha_1 \dots \alpha_p} = \begin{vmatrix} d_1 x^{\alpha_1} & d_1 x^{\alpha_2} & \dots & d_1 x^{\alpha_p} \\ d_2 x^{\alpha_1} & d_2 x^{\alpha_2} & \dots & d_2 x^{\alpha_p} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ d_p x^{\alpha_1} & \dots & \dots & d_p x^{\alpha_p} \end{vmatrix}$$

The normal element to this is not a unique vector as in the case of the magnetic shell, but a unique $(n-p)$-element which may be represented

by $$\sqrt{g} \cdot \epsilon_{\alpha_1 \alpha_2 \dots \alpha_n} \cdot d\sigma^{\alpha_{n-p+1} \dots \alpha_n} \ *$$

where ϵ is a tensor of weight -1 having components $+1$, or zero, or -1.

This $(n-p)$-dimensional normal element contains the totality of permissible displacements of the charge and corresponds to $\Delta\sigma.\mathbf{n}$ of the previous paragraph. The normal element multiplied by the charge is

$$\mu \ \sqrt{g} \cdot \epsilon_{\alpha_1 \alpha_2 \dots \alpha_n} d\sigma^{\alpha_{n-p+1} \dots \alpha_n}.$$

The inner product of this tensor with the force vector $v^i = \text{grad}\ \phi\ (r)$ will be taken to represent the tensor of work, *i.e.*, the potential for the element $d\sigma^{\alpha_{n-p+1} \dots \alpha_n}$ at O.

This will be called the lower tensor potential of $d\sigma^{\alpha_{n-p+1} \dots \alpha_n}$ at O, and denoted by $dL_{\alpha_1 \alpha_2 \dots \alpha_{n-p-1}}$.

$L_{\alpha_1 \alpha_2 \dots \alpha_{n-p-1}}$, the lower tensor potential for the whole surface, is given by $\int_\sigma dL_{\alpha_1 \alpha_2 \dots \alpha_{n-p-1}}$.

On putting

$$\sqrt{g}\ \epsilon_{\alpha_1 \dots \alpha_n} . d\sigma^{\alpha_{n-p+1} \dots \alpha_n} = dP_{\alpha_1 \dots \alpha_{n-p}}$$

---

* For the Epsilon tensor see Veblen: *Invariants of Quadratic Differential Forms*, p. 26.

we have

$$L_{\alpha_1 \ldots \alpha_{n-p-1}} = \int \mu \,.\, dP_{\alpha_1 \ldots \alpha_{n-p-1} i}\, V^i$$
$$= -\int \mu \, dP_{\alpha_1 \ldots \alpha_{n-p-1} i}\, v^i$$

where v_i are the components of grad ϕ (r), the differentiation being with respect to the co-ordinates of O instead of those of Q as in V^i.

5. In this paragraph will be defined the operations necessary for deriving certain properties of the lower tensor potential.

(i)* ***The skew product of a tensor and a vector*** :—

The product

$$\frac{1}{k!} \delta^{\alpha_1 \ldots \alpha_{k+1}}_{ij \ldots l\,.\,m}\, u^i\, T^{j \ldots lm}$$

will be called the skew product of $T^{\cdots}$ and u^i and will be denoted by

$$[u^{\alpha_1}\, T^{\alpha_2 \ldots \alpha_{k+1}}].$$

Similarly the product

$$\frac{1}{k!} \delta^{ij \ldots lm}_{\alpha_1 \ldots \alpha_{k+1}}\, u_i\, T_{j \ldots lm}$$

will be denoted by

$$[u_{\alpha_1} \,.\, T_{\alpha_2 \ldots \alpha_{k+1}}].$$

(ii) † ***The Brouwer tensor*** :—

If $(X_{\alpha_1 \ldots \alpha_k})$ is a linear tensor (*i.e.*, covariant and alternating in pairs of suffixes),

$$Z_{\alpha_1 \ldots \alpha_{k-1}} = X_{\alpha_1 \ldots \alpha_{k-1}}{}^{\nu}{}_{0}\, (\nu)$$

where (ν) denotes covariant differentiation with respect to x_ν is called the Brouwer tensor of $(X_{\alpha_1 \ldots \alpha_k})$.

and the operation is symbolically written as

$$Z_{\alpha_1 \ldots \alpha_{k-1}} = D''\,(X_{\alpha_1 \ldots \alpha_k}).$$

* $\delta^{\cdots}_{\cdots}$ is the generalised Kronecker delta with components $+1$, -1 or 0 See Veblen *Invariants of Quadratic Differential Forms*, Pages 3 and 26.

† The notation is from Weitzenbock: *Invariantentheorie* : for proof of $D''\, D''\, X_{\alpha_1 \ldots \alpha_k} \equiv 0$, see page 396.

The operation performed twice in succession gives a new tensor whose components all vanish,

i.e., $$D'' D'' (X_{\alpha_1 \ldots \alpha_k}) \equiv 0.$$

6. We shall now obtain the following alternative expression for the lower tensor potential

$$L_{\alpha_1 \ldots \alpha_{n-p-1}} = -\frac{1}{p+1} \int \mu . \sqrt{g} . \epsilon_{\alpha_1 \ldots \alpha_n} [v^{\alpha_{n-p}}{}' . d\sigma^{\alpha_{n-p+1} \ldots \alpha_n}]$$

for

$$dL_{\alpha_1 \ldots \alpha_{n-p-1}} = -\mu . dP_{\alpha_1 \ldots \alpha_{n-p-1}}{}^{i}\, v^i$$

$$= -\mu \sqrt{g} . \epsilon_{\alpha_1 \ldots \alpha_{n-p} ij \ldots q.s.} \, d\sigma^{i \ldots s} \, v^{\alpha_{n-p}}$$

$$= -\mu \sqrt{g}\, p! \,\{ \text{terms written without repetition} \}$$

$$= -p \sqrt{g} \frac{p!}{(p+1)!} \epsilon_{\alpha_1 \ldots \alpha_n} \left[v^{\alpha_{n-p}} d\sigma^{\alpha_{n-p+1} \ldots \alpha_n} \right]$$

7. $L_{\alpha_1 \ldots \alpha_{n-p-1}} = -D'' \int \mu \, \phi(r) \, dP_{\alpha_1 \ldots \alpha_{n-p}}.$

Proof:—Since the differentiation is with respect to co-ordinates of O and the integration is with respect to the surface, the two operations can be permuted. Therefore the expression on the right side is equal to

$$-\int D'' . (\mu \phi(r) \, dP_{\alpha_1 \ldots \alpha_{n-p}})$$

$$= -\int \mu v^i \, dP_{\alpha_1 \ldots \alpha_{n-p-1}\, i}$$

$$= L_{\alpha_1 \ldots \alpha_{n-p-1}}.$$

8. In the special case in which

$$p = n-1, \ \phi(r) = \frac{1^*}{r^{n-2}} \quad i.e. \quad f(r) = \frac{1}{r^{n-1}},$$

we have the exact analogue of the magnetic shell. The normal element is a unique vector and the lower potential is consequently a scalar.

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* A polynomial in r which satisfies the Laplace Equation $\nabla^2 \phi(r) = 0$ in n dimensions is proportional to $\frac{1}{r^{n-2}}$.

$$\psi = -\int \mu \, dP_i \, v_i$$
$$= -\int \mu \,.\, \sqrt{g} \,.\, \epsilon_{i\alpha_2 \ldots \alpha_n} \, d\sigma^{\alpha_2 \ldots \alpha_n} \, v^i$$
$$= -\int \mu \, \sqrt{g} \,.\, \epsilon_{i\alpha_2 \ldots \alpha_n} \, d\sigma^{\alpha_2 \ldots \alpha_n} \,.\, f(r) \, w^i$$

where w^i is a unit vector in the direction of r

$= -\frac{\mu}{r^{n-1}}$ extent of the projection of $d\sigma$ on the $(n-1)$ sphere with centre O and radius r;

$$= -\int \mu \, d\Omega$$

where $d\,\Omega$ is the extent of the sphere of unit radius intercepted by the cone joining O to the boundary of $d\sigma^{\cdots\;\cdots}$.

Hence the expression has the same form as in three dimensions where $d\Omega$ becomes the solid angle at O

It is evident from this that the lower potential of a closed $(n-1)$ surface of constant strength μ is a constant, being the extent of the unit $(n-1)$ sphere multiplied by μ.

9.* The Stokes Tensor :—

Let $x_{\alpha_1 \ldots \alpha_k}$ be a linear tensor.

Then $$X_{\alpha_1 \ldots \alpha_{k-1}} = \frac{1}{k!} \delta^{ij \ldots lm}_{\alpha_1 \ldots \alpha_{k+1}} \frac{\partial x_{j \ldots lm}}{\partial x_i}$$

is called the Stokes tensor of $x_{\alpha_1 \ldots \alpha_k}$ and is symbolically written

$$X_{\alpha_1 \ldots \alpha_{k+1}} = D' \left(x_{\alpha_1 \ldots \alpha_k} \right).$$

Also $D'D' \left(x_{\alpha_1 \ldots \alpha_k} \right) \equiv 0$, the expression written fully cancelling out in the double summation.

10. The Higher Tensor Potential :—

The idea of vector potential of a double layer will now be generalised and some of its properties given.

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* Except that the expression for the Stokes tensor in the definition is written in a compact form by the use of Kronecker deltas, the notation is from Weitzenbock: *Invariantentheorie*, p. 191.

In Euclidean S_3, for a magnetic shell of constant strength μ, the vector potential is defined as $\mu \int \frac{dS}{r}$ where dS is the directed line element of the boundary, the integral being taken over the whole of the closed boundary of the shell.

Let the closed boundary of the p-dimensional surface be denoted by τ. Taking $\phi(r)$ as the potential function as before, the potential (corresponding to the vector-potential of the shell) may be defined as $\mu \int_\tau \phi(r)\, d\tau^{\alpha_1 \ldots \alpha_{p-1}}$, where μ is constant and the integral is taken over the boundary.

Assuming that the boundary is two-sided, *i.e.*, has two distinct opposite senses of description, we can express the potential in terms of the surface of the double layer. Let $A_{\alpha_1 \ldots \alpha_{p-1}}$ be an arbitrary linear tensor. Consider the integral

$$\mu A_{\alpha_1 \ldots \alpha_{p-1}} \int_{(\tau)} \phi(r)\, d\tau^{\alpha_1 \ldots \alpha_{p-1}}$$

$$= \mu \int A_{\alpha_1 \ldots \alpha_{p-1}} \phi(r)\, d\tau^{\alpha_1 \ldots \alpha_{p-1}}$$

since $A_{\alpha_1 \ldots \alpha_{p-1}}$ is arbitrary.

This may be written

$$\mu \int_{(\sigma)} D'\left(A_{\alpha_1 \ldots \alpha_{p-1}} \phi(r)\right) d\sigma^{\alpha_1 \ldots \alpha_p}$$

by the generalised theorem of Stokes, the boundary being two-sided;

$$= \mu \int_{(\sigma)} [v_{\alpha_1} A_{\alpha_2 \ldots \alpha_p}]\, d\sigma^{\alpha_1 \ldots \alpha_p}.$$

Since A... is arbitrary

$$= p\,\mu \,.\int_{(\sigma)} A_{\alpha_2 \ldots \alpha_p} \,.\, v_i\, d\sigma^{i\alpha_2 \ldots \alpha_p}$$

$$(-1)^{p-1} A_{\alpha_1 \ldots \alpha_{p-1}} \,.\, p \,.\, \mu \,.\int v_i\, d\sigma^{\alpha_1 \ldots \alpha_{p-1} i}$$

changing the dummy suffixes.

$$\therefore \quad \mu \int \phi(r)\, d\tau^{\alpha_1 \ldots \alpha_{p-1}} = (-1)^{p-1} p.\mu. \int v_i\, d\sigma^{\alpha_1 \ldots \alpha_p}.i$$

Again from the last expression can be derived a covariant tensor by contraction with the Epsilon tensor and multiplication by $\sqrt{g}$. This

tensor multiplied by $(-1)^{n-p}$* will be called the Higher tensor potential and denoted by $H_{\alpha_1 \ldots \alpha_{n-p+1}}$ and is equal to

$$\int (-1)^{n-p} (-1)^{p-1} . p . \mu . \sqrt{g} . \epsilon_{\alpha_1 \ldots \alpha_{n-p+1} j \ldots qs} . v_i . d\sigma^{j \ldots qs . i}.$$

11. $H_{\alpha_1 \ldots \alpha_{n-p+1}} = D' \int \mu \, \phi(r) \, dP_{\alpha_1 \ldots \alpha_{n-p}}$

Proof:—

$$H_{\alpha_1 \ldots \alpha_{n-p+1}} = \int (-1)^{n-1} \mu . \sqrt{g} . p . \epsilon_{\alpha_1 \ldots \alpha_{n-p+1} j \ldots qs} . v_i . d\sigma^{j \ldots si}$$

$$= \int (-1)^{n-1} \mu . \sqrt{g} . p . (-1)^{n-1} ! \ (p-1)! \ \{\text{terms in the}$$

skew product of v_i and $(\epsilon_{\alpha_1 \ldots \alpha} d\sigma_{\alpha_{n-p+1} \ldots \alpha_n}$ written without repetition)$\}$

$$= \int \mu . \frac{p . (p-1)!}{p!} [v_{\alpha_1} [dP_{\alpha_2 \ldots \alpha_{n-p+1}}]$$

$$= \int D' \mu \phi(r) \, dP_{\alpha_1 \ldots \alpha_{n-p}} \quad \text{since } \mu \text{ is constant,}$$

$$= D' \int \mu \phi(r) \, dp_{\alpha_1 \ldots \alpha_{n-p}}$$

since the order of integration and differentiation can be changed.

12. In what follows μ is taken equal to 1 for convenience.

It will be shown that the Higher tensor potential of a closed p-dimensional surface of double distribution (of constant strength, μ is here put as equal to 1) at a point outside it is zero.

For, consider a two-sided $(p-1)$-dimensional closed boundary (*i.e.*, a boundary with two distinct senses of description). Let σ be an open p-dimensional surface fitting it. The transformations which transform the boundary into itself transform σ into σ' with the same boundary. They either preserve the sense of the boundary or not according as the Jacobian of the transformation preserves its sign or not. The non-singular transformations which transform the boundary into itself, preserving the sense at the same time, fall into two classes corresponding to the two senses of description of the boundary. These

* Multiplication by $(-1)^{n-p}$ facilitates further calculations.

two classes of transformations give two sets of open p-dimensional surfaces (σ) and (Σ) fitting the closed boundary.

The sum of two surfaces one from each of these classes (σ) and (Σ) having the same boundary constitutes a closed p-dimensional surface. The Higher tensor potential due to such a closed surface is obviously zero, for it consists of two parts, each the negative of the other, correrponding to the two senses of description of the boundary.

13. The Field :—

Two covariant potentials $L_{\alpha_1 \ldots \alpha_{n-p-1}}$ and $H_{\alpha_1 \ldots \alpha_{n-p-1}}$ have been obtained for the double layer. We can derive two tensors of order $n - p$ from these two potentials, namely,

$$ {}^1F_{\alpha_1 \ldots \alpha_{n-p}} = - D' (L_{\alpha_1 \ldots \alpha_{n-p-1}}) $$

and

$$ {}^2F_{\alpha_1 \ldots \alpha_{n-p}} = D'' (H_{\alpha_1 \ldots \alpha_{n-p+1}}). $$

These two processes correspond to the two ways of deriving the field (i) as the negative of the gradient of a scalar potential and (ii) as the curl of a vector potential.* Taking either of them as a measure of the field, we shall next inquire into the conditions necessary and sufficient for the equivalence of the field derived in the two ways.

14. (i) *Necessary conditions for* ${}^1F_{\alpha_1 \ldots \alpha_{n-p}} = {}^2F_{\alpha_1 \ldots \alpha_{n-p}}$

Suppose $${}^1F_{\alpha_1 \ldots \alpha_{n-p}} = {}^2F_{\alpha_1 \ldots \alpha_{n-p}}$$

$$ \begin{aligned} D'' ({}^1F_{\alpha_1 \ldots \alpha_{n-p}}) &= D'' ({}^2F_{\alpha_1 \ldots \alpha_{n-p}}) \\ &= D'' D'' \int \mu\phi(r)\, dP_{\alpha_1 \ldots \alpha_{n-p}} \\ &= 0. \end{aligned} $$

Similarly $$ \begin{aligned} D' ({}^2F_{\alpha_1 \ldots \alpha_{n-p}}) &= D' ({}^1F_{\alpha_1 \ldots \alpha_{n-p}}) \\ &= D'D' \int \mu\phi(r)\, dP_{\alpha_1 \ldots \alpha_{n-p}} \\ &= 0. \end{aligned} $$

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* See paragraph 15.

Thus $$D''\,({}^1F_{\alpha_1 \cdots \alpha_{n-p}}) = 0$$

and $$D'\,({}^2F_{\alpha_1 \cdots \alpha_{n-p}}) = 0$$

are the necessary conditions.

(ii) *Sufficient condition. for* ${}^1F_{\alpha_1 \cdots \alpha_{n-p}} = {}^2F_{\alpha_1 \cdots \alpha_{n-p}}$.

If the potential function $\phi(r)$ satisfies the Laplace Equation $\nabla^2\phi(r)=0$ at points in free space, *i.e.*, at points outside the double layer, then

$${}^1F_{\alpha_1 \dots \alpha_{n-p}} = {}^2F_{\alpha_1 \cdots \alpha_{n-p}}.$$

Proof :—Since in rectangular co-ordinates

$$\nabla^2 \phi(r) = 0$$

$$\therefore \quad \nabla^2 \int \mu\phi(r)\, dP_{\alpha_1 \cdots \alpha_{n-p}} = 0$$

because the change of the order integration and differentiation makes integrand vanish.

Therefore, the tensor vanishes in all co-ordinate systems.

Therefore its equivalent* in other systems, *viz.*,

$$(D''D' - D'D'') \int \mu\phi(r)\, dP_{\alpha_1 \dots \alpha_{n-p}} = 0,$$

i.e., $$D''H_{\alpha_1 \cdots \alpha_{n-p+1}} = -D'\,L_{\alpha_1 \cdots \alpha_{n-p-1}}$$

$${}^2F_{\alpha_1 \cdots \alpha_{n-p}} = {}^1F_{\alpha_1 \cdots \alpha_{n-p}}.$$

15. The derivation of the field in the above twofold manner (i) as the Stokes tensor of Lower potential and (ii) as the Brouwer tensor of the Higher potential is applicable to the case of the magnetic shell in S_3.

To show that the usual equation, *viz.*,

$$\text{Field} = -\operatorname{grad}\psi = \operatorname{curl}(P_i)$$

can be written in the above form we proceed as follows :—

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* For the equivalent form ∇^2 see Weitzenbock *Invariantentheorie*, p. 397.

Using rectangular co-ordinates, we need not distinguish between the covariant and contravariant components of a vector ; also $\sqrt{g} = 1$.

$$\therefore \quad P_1 = P^1 = \epsilon_{123} P^1 = \epsilon_{231} P^1$$

and may be written $\quad R_{23}$ (for $\epsilon_{123} = \epsilon_{231} = 1$).

Similarly $P_2 = R_{31}$ and $P_3 = R_{12}$

$$\therefore \qquad \text{Curl } (P_i) = \left\{ \left(\frac{\partial P_3}{\partial x_2} - \frac{\partial P_2}{\partial x_3}\right), \text{ etc.} \right\}$$

$$= \left\{ \left(\frac{\partial R_{12}}{\partial x_2} - \frac{\partial R_{31}}{\partial x_3}\right), \text{ etc.} \right\}$$

$$= \left\{ \left(\frac{\partial R_{12}}{\partial x_2} + \frac{\partial R_{13}}{\partial x_3}\right), \text{ etc.} \right\}$$

$$= D'' (R_{ij}).$$

Therefore, the equation can be written in the form

$$\text{Field} = -\text{grad } \psi = D'' (R_{ij})$$

$$= - D' \psi = D'' (R_{ij}).$$

Hence the usual equation is a particular case of the more general one

$$\text{Field} = - D' (L_{\alpha_1 \ldots \alpha_{n-p-1}}) = D'' (H_{\alpha_1 \ldots \alpha_{n+p+1}}).$$

16. Finally reference must be made to a paper of L. E. J. Brouwer* on ' Polydimensional Vector Distribution's. The results obtained here are in agreement with the ones given in his paper. Two such outstanding results will be given here.

(i) He defines an operation ∇^2 which corresponds to D''. Besides deriving the field as the gradient of a scalar potential, he derives it also as the ∇^2 of a plani-vector potential, *i.e.*, the D'' of a tensor of

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* The paper referred to is 'Polydimensional Vector Distributions' by E. L. J. Brouwer in the *Proceedings of Sciences, Amsterdam*, Vol. 9, 1, p. 66.

second order The special cases treated of in his paper correspond to the case

$$p = n - 1 \text{ and } \phi(r) = \frac{1}{r^{n-2}}.$$

(ii) If the field is derived as the Stokes tensor of the potential, the potential itself is derivable as Brouwer tensor, and *vice versa.*

Here also the same results hold, for

$$^{1}F_{\alpha_1 \ldots \alpha_{n-p}} = -D'(L_{\alpha_1 \ldots \alpha_{n-p-1}})$$

while $$L_{\alpha_1 \ldots \alpha_{n-p-1}} = -D'' \int \mu\phi(r)\, dP_{\alpha_1 \ldots \alpha_{n-1}};$$

and $$^{2}F_{\alpha_1 \ldots \alpha_{n-p}} = D''(H_{\alpha_1 \ldots \alpha_{n-p+1}})$$

while $$H_{\alpha_1 \ldots \alpha_{n-p+1}} = D' \int \mu\phi(r)\, dP_{\alpha_1 \ldots \alpha_{n-p}}.$$

The field considered here is that due to a definite double layer without any restriction however on the value of p and hence without any restriction on the order of the potentials such as always being a scalar and second order linear tensor.

ON NUMBERS IN MEDIAL PROGRESSION

By Hansraj Gupta,
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1. A sequence of numbers

$$a_1, a_2, a_3, \ldots a_n, \ldots$$

is said to be in *Medial Progression* (*M.P.*) if

$$a_r = a_{r-1} + a_{r-2} \qquad [r \geqslant 3]. \qquad \ldots \quad (1)$$

Throughout this paper which develops properties of such a progression, the numbers a_n will be supposed to be positive or negative integers.

2. (*a*) *To find the sum of n numbers in medial progression.*

Let $\qquad S_n = a_1 + a_2 + a_3 + a_4 + \ldots + a_n;$

also $\qquad S_n = \qquad a_1 + a_2 \qquad + \ldots + a_{n-2} + a_{n-1} + a_n.$

Subtracting,

$$\begin{aligned} 0 &= a_1 + a_2 + a_2 + a_3 + \ldots\ldots + a_{n-1} - a_{n-1} - a_n \\ &= a_2 + (a_1 + a_2 + a_3 + \ldots\ldots + a_{n-2}) - a_n \\ &= a_2 + S_{n-2} - a_n. \end{aligned}$$

$\therefore \qquad S_{n-2} = a_n - a_2.$

$\therefore \qquad S_n = a_{n+2} - a_2.$

But $\qquad a_{n+2} = 2a_n + a_{n-1}.$

$\therefore \qquad S_n = 2a_n + a_{n-1} - a_2.$

(*b*) *To express any term of the series in terms of a_1 and a_2.*

We have

$$\begin{aligned} a_3 &= a_2 + a_1, \\ a_4 &= a_3 + a_2 = 2a_2 + a_1, \\ a_5 &= 3a_2 + 2a_1, \\ a_6 &= 5a_2 + 3a_1, \\ a_7 &= 8a_2 + 5a_1, \\ &\ldots \quad \ldots \quad \ldots \end{aligned}$$

Notice that the co-efficients of a_1 and a_2 are themselves in M.P. this being the series 1, 1, 2, 3, 5, 8, ... , [called the ***Natural Medial Progression***].

If N_r denote the rth term of this progression, we obtain

$$a_r = N_{r-1}\, a_2 + N_{r-2}\, a_1 \qquad \dots \quad (2)$$

Remembering that a medial progression is a recurring series whose scale of relation is $1 - x - x^2$, we get

$$N_r = \frac{1}{\sqrt{5}} \left[\left\{ \left(\frac{\sqrt{5}+1}{2}\right)^r - (-1)^r \left(\frac{\sqrt{5}-1}{2}\right)^r \right\} \right] \dots \quad (3)$$

3. Regarding a_s and a_{s+1} as the first two terms of the series we get

$$a_{r+s-1} = N_{r-1}a_{s+1} + N_{r-2}a_s = N_r a_s + N_{r-1}a_{s-1}. \qquad \dots \quad (4)$$

As particular cases, we have

$$N_{r+s-1} = N_{r-1}N_{s+1} + N_{r-2}N_s = N_r N_s + N_{r-1}N_{s-1} \quad \dots \quad (5)$$

$\therefore \qquad N_{2r-1} = N_r^2 + N_{r-1}^2 \qquad \dots \quad (6)$

and $\qquad N_{2r} = N_r N_{r+1} + N_{r-1}N_r = N_r\,(N_{r+1} + N_{r-1}). \qquad \dots \quad (7)$

Hence $\qquad N_{2r} \equiv 0 \quad [\text{mod. } N_r]$.

This result is a particular case of a more general result which w obtain by putting $s \equiv 1$ [mod. r] in (5).

Let $\qquad s = 1 + kr;$

then $\qquad N_{(k+1)r} = N_r N_{kr+1} + N_{r-1}N_{kr}.$

Hence, if $N_{kr} \equiv 0$ [mod. N_r], then $N_{(k+1)r} \equiv 0$ [mod. N_r].

But $\qquad N_{2r} \equiv 0$ [mod. N_r].

Hence by induction $\qquad N_{kr} \equiv 0$ [mod. N_r] $\qquad \dots \quad (8)$

or $\qquad N_n \equiv 0$ [mod. N_m], if $n \equiv 0$ [mod. m]. $\qquad \dots \quad (8a)$

4. *If p be a prime, N_p must be prime to every preceding term of the series.*

If not, let N_r be the least term not prime to N_p and $r < p$.

Suppose $$p = kr + q, \qquad (q < r).$$

Now $$N_p \equiv N_q N_{kr+1} + N_{q-1} N_{kr};$$

also $$N_{kr} \equiv 0 \text{ [mod. } N_r].$$

Again N_{kr+1} and N_{kr}, being consecutive terms of a Natural Medial Progression, are mutually prime.

$$\therefore \qquad N_{kr+1} \not\equiv 0 \text{ [mod. } N_r] \text{ and } \{ N_{kr+1}, N_r \} = 1.$$

$$\therefore \qquad \{ N_p, N_r \} = \{ N_p, N_q \}, \qquad q < r.$$

This is contrary to the supposition.

Hence N_p must be prime to every preceding term of the series. Such terms may be called the '*Natural Medial Primes*.'

We have incidently proved here that

$$\{ N_r, N_q \} = 1, \text{ if } \{ r, q \} = 1 \text{ or } 2 \qquad \ldots \quad (9)$$

and $$\{ N_r, N_q \} = N_h, \text{ where } \{ r, q \} = h. \qquad \ldots \quad (10)$$

From the preceding results it follows that

$$N_n \equiv 0 \qquad \text{[mod. } N_{p_1} N_{p_2} N_{p_3} \ldots\ldots]$$

where $p_1, p_2, p_3, \ldots$ are the prime factors of n. ... (11)

5. *To find* $\underset{n \to \infty}{\text{Lt}} \dfrac{a_{n-1}}{a_n}$.

We have $$\frac{a_{n-1}}{a_n} = \frac{1}{a_n/a_{1-n}} = \frac{1}{1+} \frac{a_{n-2}}{a_{n-1}} = \frac{1}{1+} \frac{1}{1+} \frac{a_{n-3}}{a_{n-2}}$$

$$= \frac{1}{1+} \frac{1}{1+} \frac{1}{1+} \frac{1}{1+} \cdots \cdots \frac{1}{1+} \frac{a_1}{a_2}$$

Therefore, if $$\text{Lt} \frac{a_{n-1}}{a_n} = x, \quad \text{then } x = \frac{1}{1+x}$$

or $$x^2 + x - 1 = 0.$$

Solving, we get $$x = \frac{\sqrt{5}-1}{2}.$$

$$\therefore \qquad \underset{n\to\infty}{\mathrm{Lt}} \frac{a_{n-1}}{a_n} = \frac{\sqrt{5}-1}{2}. \qquad \ldots \quad (12)$$

Also $$\underset{n\to\infty}{\mathrm{Lt}} \frac{a^2_{n-1}}{a_n a_{n-2}} = \underset{n\to\infty}{\mathrm{Lt}} \frac{a_{n-1}}{a_n} \Big/ \underset{n\to\infty}{\mathrm{Lt}} \frac{a_{n-2}}{a_{n-1}} = 1.$$

But $$a_n = a_{n-1} + a_{n-2}.$$

Hence in the limit a_n has been so divided into two parts that the product of the whole and one of the parts is equal to the square of the other part.

Hence the name of the present series. It may be observed that $(-1)^n \left[a_n^2 - a_{n+1} \cdot a_{n-1}\right]$ is a constant of the progression and may be called its ***Medial difference.***

6. Keeping in mind the relation $a_n = a_{n-1} + a_{n-2}$ or $a_{n-2} = a_n - a_{n-1}$, we can continue a M. P. backwards through terms of negative order.

Thus
$$a_0 = a_2 - a_1,$$
$$a_{-1} = a_1 - a_0 = -a_2 + 2a_1,$$
$$a_{-2} = a_0 - a_{-1} = 2a_2 - 3a_1,$$
$$a_{-3} = a_{-1} - a_{-2} = -3a_2 + 5a_1,$$

As a matter of fact, we have

$$N_0 = 0,\ N_{-1} = 1,\ N_{-2} = -1,\ N_{-3} = 2,\ \ldots \ \ldots$$

and, in general $N_{-2p} = -N_{2p}$, and $N_{-2p+1} = N_{2p-1}$,

so that $$N_{-r} = (-1)^{r+1} N_r \qquad \ldots \quad (13)$$

Hence
$$a_{-r} = N_{-r-1} a_2 + N_{-r-2} a_1$$
$$= (-1)^r \{ N_{r+1} a_2 - N_{r+2} a_1 \}, \qquad \ldots \quad (14)$$

7. *Given a_p and a_q, two terms of an M.P., to find a_r.*

We have,
$$a_p = N_{p-1} a_2 + N_{p-2} a_1,$$
$$a_q = N_{q-1} a_2 + N_{q-2} a_1,$$
$$a_r = N_{r-1} a_2 + N_{r-2} a_1.$$

Eliminating a_1, a_2 between these three equations, we get

$$\begin{vmatrix} a_p & N_{p-1} & N_{p-2} \\ a_q & N_{q-1} & N_{q-2} \\ a_r & N_{r-1} & N_{r-2} \end{vmatrix} = 0 \qquad \ldots \quad (15)$$

This gives a_r.

Simplifying, we get

$$(-1)^q a_r N_{p-q} + (-1)^r a_p N_{q-r} + (-1)^p a_q N_{r-p} = 0$$

or

$$\Sigma (-1)^q a_r N_{p-q} = 0. \qquad \ldots \quad (16)$$

8. Let $a_1, a_2, a_3, \ldots, a_n \ldots$ and $b_1, b_2, b_3, \ldots, b_n \ldots$ be two sets of numbers in M. P.

Now,

$$a_p = N_{p-1} a_2 + N_{p-2} a_1,$$
$$b_q = N_{q-1} b_2 + N_{q-2} b_1,$$
$$a_q = N_{q-1} a_2 + N_{q-2} a_1,$$
$$b_p = N_{p-1} b_2 + N_{p-2} b_1.$$

$$\therefore \quad a_p b_q - a_q b_p = (N_{p-1} N_{q-2} - N_{q-1} N_{p-2}) (a_2 b_1 - a_1 b_2)$$
$$= (-1)^{q-1} N_{p-q} . (a_2 b_1 - a_1 b_2). \qquad \ldots \quad (17)$$

From this we can easily deduce that

$$a_p^2 - a_{p-1} a_{p+1} = (-1)^{p-2} (a_2^2 - a_1 a_3).$$

In general,

$$a_p b_q - a_{p-m} b_{q+m} \equiv 0 \quad [\text{mod. } N_m] \qquad \ldots \quad (18)$$

Result (17) can be conveniently put in the form

$$\begin{vmatrix} a_p & a_q \\ b_p & b_q \end{vmatrix} = (-1)^{q-1} . N_{p-q} \begin{vmatrix} a_2 & a_1 \\ b_2 & b_1 \end{vmatrix} \qquad \ldots \quad (19)$$

$$\therefore \quad \begin{vmatrix} a_{p+m} & a_{q+m} \\ b_p & b_q \end{vmatrix} = (-1)^{q-1} . N_{p-q} \begin{vmatrix} a_{m+2} & a_{m+1} \\ b_2 & b_1 \end{vmatrix} \ldots \quad (19a)$$

Further we have

$$a_{n+k}\mathrm{N}_{-k}-a_{n-k}\mathrm{N}_k = \begin{vmatrix} a_{n+k} & a_{n-k} \\ \mathrm{N}_k & \mathrm{N}_{-k} \end{vmatrix}$$

$$= (-1)^{k-1}\cdot \mathrm{N}_{2k} \begin{vmatrix} a_{n+2} & a_{n+1} \\ \mathrm{N}_2 & \mathrm{N}_1 \end{vmatrix}$$

$$= (-1)^{k-1}\cdot \mathrm{N}_{2k}\cdot a_n = (-1)^{k-1} a_n \mathrm{N}_k(\mathrm{N}_{k+1}+\mathrm{N}_{k-1}).$$

$$\therefore \quad a_{n+k}+(-1)^k a_{n-k} = a_n(\mathrm{N}_{k+1}+\mathrm{N}_{k-1}) \qquad \dots \quad (20)$$

Now,
$$a_{n-k} = a_n \mathrm{N}_{-k+1} + a_{n-1}\mathrm{N}_{-k}$$
$$= (-1)^k \{ a_n \mathrm{N}_{k-1} - a_{n-1}\mathrm{N}_k \}.$$

$$\therefore \quad a_{n+k}-(-1)^k a_{n-k} = a_n(\mathrm{N}_{k+1}+\mathrm{N}_{k-1})-2(a_n\mathrm{N}_{k-1}-a_{n-1}\mathrm{N}_k)$$
$$= a_n(\mathrm{N}_{k+1}-\mathrm{N}_{k-1})+2a_{n-1}\mathrm{N}_k$$
$$= \mathrm{N}_k(a_n+2a_{n-1}) = \mathrm{N}_k(a_{n+1}+a_{n-1}). \quad \dots \quad (21)$$

9. ***Auxiliary series.***

Consider the series $\Sigma \mathrm{A}_n$, such that

$$\mathrm{A}_n = a_{n+1}+a_{n-1}, \qquad n \geqslant 1 \qquad \dots \quad (22)$$

where $a_0, a_1, a_2, \dots a_{n-1}, a_n, a_{n+1}, \dots$ are in M.P.;

we have
$$\mathrm{A}_n + \mathrm{A}_{n+1} = (a_{n+1}+a_{n-1})+(a_{n+2}+a_n)$$
$$= (a_{n+1}+a_{n+2})+(a_{n-1}+a_n)$$
$$= a_{n+3}+a_{n+1} = \mathrm{A}_{n+2}.$$

Hence $\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3, \dots, \mathrm{A}_n, \dots$ are in M.P.

$\Sigma \mathrm{A}_n$ is called the *auxiliary series corresponding to* Σa_n. The auxiliary series of the Natural Medial Progression is of special importance. Its rth term will be denoted by M_r.

We shall have
$$\mathrm{M}_r = \mathrm{N}_{r+1}+\mathrm{N}_{r-1}. \qquad \dots \quad (23)$$

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Notes and Questions.

Lecture Notes.

[The Editors will be glad to publish under this heading brief suggestions likely to be of help in the class-room].

1. The Dimensionality of Constants in Formulae in Analytic Geometry.

I have often found the concept of the dimension of constants in formulae in analytical geometry helpful in checking their accuracy. Thus, if we conceive of x, y the non-homogeneous co-ordinates of a point in the metrical plane, as actual distances and, therefore, of dimension 1 in length and

$$lx + my + n = 0.$$

as the equation of a line, it is clear that if n be a number, l and m must be of dimension -1 in length. Thus

$$(lx_1 + my_1 + n) / \sqrt{l^2 + m^2}$$

is of dimension 1 in length, which is proper for the distance of a point from a line. A further condition that has to be fulfilled is that the formula should be homogeneous in l, m, n and of dimension zero, since the lines (l, m, n) and (kl, km, kn) are identical.

As another example, consider the area of the conic given by

$$al^2 + 2hlm + bm^2 + 2gln + 2fmn + cn^2 = 0 \qquad \dots \quad (1)$$

Since l and m are of dimension -1, and n of dimension 0 in length if we take c as an absolute constant, a, h and b are of dimension 2 and g and f of dimension 1 in length. The area A of the conic vanishes if

$$\begin{vmatrix} a & h & b \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

when (1) represents a pair of points (one of the principal axes is finite and the other zero) and becomes infinite if the conic is a parabola, *i.e.*,

if $c=0$. It is found that these are the only cases when the area vanishes or is infinite so that we have

$$A^2 = k \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}^r \div c^s$$

where k, r, s are absolute constants. Remembering that A^2 must be

(i) homogeneous and of degree zero in a, b, c, f, g, h,

and (ii) of dimension 4 in length, since it represents the square of an area,

we easily find $r = 1$ and $s = 3$. Applying the formula to the circle $l^2 + m^2 = n^2$ of unit radius we have $k = \pi^2$.

It is possible to obtain even complicated formulae from considerations like the above and the concept of the dimensionality of the constants and I have obtained in this way a formula for the lengths of the axes of a quadric locus or envelope in n-dimensional Euclidean space,* and for the product of the lengths of normals from a point to a curve.† In these, the cases where some of the lines or curves are specially related to the circular points require very careful examination.

The use of the concept of dimensionality to check the accuracy of a formula obtained already is however a much simpler matter, and should be helpful in detecting errors which might otherwise be unnoticed.

A. NARASINGA RAO.

2. The Conic in Analytical Geometry.

(2.1) *To find the asymptotes of*

$$S(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \quad \ldots \quad (1)$$

Let $(x_1 y_1)$ be a point on one of the asymptotes. On transferring the origin to $(x_1 y_1)$, the equation becomes

$$ax^2 + 2hxy + by^2 + 2\xi_1 x + 2\eta_1 y + S_1 = 0 \quad \ldots \quad (2)$$

* *Tohoku Mathematical Journal*, Sept. 1929.
† Mathematical Gazette, Vol. 14, p. 140.

where $\xi_1 = ax_1 + hy_1 + g$ and $\eta_1 = hx_1 + by_1 + f$.
Now (2) must be of the form

$$(lx + my)(l'x + m'y + n') + k = 0$$

that is, $$(lx + my)(l'x + m'y) + n(lx + my) + k = 0. \quad \dots \ (3)$$
Comparing (3) and (2), we find that $ax^2 + 2hxy + by^2$ must have $\xi_1 x + \eta_1 y$ for a factor.

In other words

$$a\eta_1^2 - 2h\,\xi_1\eta_1 + b\xi_1^2 = 0. \quad \dots \ (4)$$

Suppressing the suffixes, (4) represents the equation of the asymptotes.

(2.2). *To find the foci of the conic* (1).

On transferring the origin to the focus $(x_1 y_1)$ we have the equation (2) which must be of the form

$$\lambda(x^2 + y^2) + (lx + my + n)^2 = 0. \quad \dots \ (5)$$

Comparing coefficients in (2) and (5), we have

$$\begin{aligned} a &= \lambda + l^2, & \xi_1 &= ln, \\ b &= \lambda + m^2, & \eta_1 &= mn, \\ h &= lm, & S_1 &= n^2, \end{aligned}$$

which determine the foci;

whence $$\frac{\xi_1^2 - \eta_1^2}{a - b} = \frac{\xi_1\eta_1}{h} = S_1.$$

Also, the directrix $lx + my + n = 0$ is given by

$$\xi_1 x + \eta_1 y + S_1 = 0.$$

B. B. BAGI.

3. On Tannery's Theorem for Infinite Products.

The usual mode of treating infinite products in most text-books dealing with the topic, is through the medium of the logarithmic function. The introduction of multiple valued functions may be avoided by the use of Euler's Identity. In this note, I derive Tannery's Theorem on Infinite Products from the corresponding theorem for infinite series.*

* *Vide* Bromwich: *Infinite Series*, pp. 123-124.

THEOREM : "*If p increases to infinity with n and*

$$\underset{n \to \infty}{\mathrm{Lt}}\ v_r(n) = w_r,$$

$$\underset{n \to \infty}{\mathrm{Lt}}\ P_n(n) \equiv \underset{n \to \infty}{\mathrm{Lt}}\ [1 + v_1(n)]\,[1 + v_2(n)]\ \ldots\ [1 + v_p(n)]$$

$$= (1 + w_1)(1 + w_2)(1 + w_3) \ldots \text{ to } \infty.$$

provided that

$$|\,v_r(n)\,| < M_r$$

where M_r *is independent of n and* $\Sigma\, M_r$ *is convergent.*

Consider the sum

$$[1 + v_1(n)] + [1 + v_1(n)]\,v_2(n) + [1 + v_1(n)]\,[1 + v_2(n)]\,v_3(n) + \ldots \text{ to } p \text{ terms.} \quad \ldots \quad (1)$$

First, we observe that the terms respectively tend to

$$(1 + w_1),\ (1 + w_1)\,w_2,\ (1 + w_1)(1 + w_2)\,w_3,\ \ldots$$

Secondly their moduli are less than

$$(1 + M_1),\quad (1 + M_1)\,M_2,\quad (1 + M_1)(1 + M_2)\,M_3,\ \ldots$$

each of which is independent of n.

Lastly the series

$$(1 + M_1) + (1 + M_1)\,M_2 + (1 + M_1)(1 + M_2)\,M_3 + \ldots \text{ to } \infty$$

converges, since the sum of its first p terms is evidently equal to

$$(1 + M_1)(1 + M_2) \ldots (1 + M_p)$$

which tends to a finite non-zero limit because $\Sigma\, M_r$ is convergent.

Thus all conditions of Tannery's Theorem for Infinite Series are fulfilled ; hence the series (1) tends to

$$(1 + w_1) + (1 + w_1)\,w_2 + (1 + w_1)(1 + w_2)\,w_3 + \ldots \text{ to } \infty. \ldots \quad (2)$$

But obviously (1) and (2) are respectively equal to

$$P_n \text{ and } \prod_1^{\infty} (1 + w_r).$$

$$\therefore \qquad P_n \to \prod_1^{\infty} (1 + w_r).$$

B. B. BAGI.

Historical Note on the Determination of Abstract Groups of Given Orders.

While the determination of all the permutation groups of given degrees was inaugurated by J. A. Serret in 1850, the determination, of all the abstract groups of given orders was inaugurated about four years later by A. Cayley. In an article published in the *Philosophical Magazine*, Vol. 7 (1854), A Cayley noted that there is one and only one abstract group of every prime order p, and he determined the two possible abstract groups of each of the orders 4 and 6, and noted that there are no other abstract groups of these orders. On the contrary, he stated, about twenty-four years later, incorrectly, in the first number of the *American Journal of Mathematics*, page 51, that there are three groups of order 6 and that the number of the abstract groups of order n is equal to the number of the permutation groups of degree n, while it is actually equal to the number of the regular permutation groups of this degree. In the former of these two articles he also noted the elementary but very important theorem that a necessary and sufficient condition that two operators of order 2 are commutative is that their product is of this order.

The first article of A. Cayley on the Theory of Groups is noteworthy not only because it involves the earliest complete enumeration of the possible abstract groups of given orders but also because it includes the earliest system of postulates of the mathematical term finite group. This system is not clearly formulated here by him but it is practically equivalent to the system given later (1882) by H. Weber and widely adopted by other writers. As regards the listing of all the possible abstract groups of a given order an article published about five years later by A. Cayley in the *Philosophical Magazine*, Volume 18 (1859), page 34, is much more important. In this article the five possible groups of order 8 are correctly determined and it is noted that there is a dihedral group of every even order. It should be observed that 8 is the lowest order for which there are more than two groups and that A. Cayley's correct determination of the comparatively large number of the groups of this order is remarkable for that time.

An interesting feature of the determination of all the possible abstract groups of given orders is that all of these groups can easily be determined for a few general orders. For instance, the facts that there are two and only two groups of order p^2, where p is any prime number, and that there are two and only two groups of order pq, (q being a prime number distinct from p) whenever one of these two primes is congruent to unity with respect to the other and only then, were established by E. Netto in his *Substitutionentheorie*, 1882. On the contrary, it is not possible to determine all the permutation groups of a general degree. In fact, little has been done towards the determination of all these groups for a general order. In the trivial case when this order is a prime number p there is obviously one and only one such permutation group of degree kp, where k is an arbitrary natural number. The cases when the order is 4 or pq were considered by G. A. Miller in the *Philosophical Magazine*, Volume 4 (1896), page 431, and in the *Bulletin of the American Mathematical Society*, Volume 2 (1896), page 332.

Not only is it possible to determine all the abstract groups of certain general orders but it is also possible in certain cases to determine all the orders for which a given number of abstract groups exist. For instance, it results directly from the proof due to G. Frobenius of the fact that every group whose order is not divisible by the square of a prime number is solvable, that a necessary and sufficient condition that there is only one group of order g is that g is the product of distinct prime numbers such that none of them is congruent to unity with respect to another one of them. The lowest composite number which is the order of but one group is therefore 15.

From the same proof it results directly that the necessary and sufficient condition that there are exactly two groups of order g is that g is either the product of distinct prime numbers such that one and only one of them is congruent to unity with respect to one and only one other such factor, or that g is divisible by the square of one and only one prime number p but by no higher power of p and that none of its prime factors is congruent to unity with respect to another such factor nor with respect to $p+2$. In particular, if there are two and only two groups of

order g and g is divisible by the square of a prime then it must be odd unless it is 4. When g is not divisible by the square of a prime number and one and only one of its prime factors p is congruent to unity with respect to another of its prime factors, and this is congruent to unity with respect to k such factors, then there are clearly exactly 2^k groups of order g and the quotient group of all these groups with respect to the characteristic sub-group of order p is cyclic. In particular, it is possible to find orders for which there are exactly 2^k groups, where k is an arbitrary natural number but it does not seem to be known whether it is possible to find orders for which the number of the possible groups is an arbitrary given prime number.

It is easy to find all the forms of g such that there are exactly three groups of order g. When g is not divisible by the square of a prime it must then obviously involve two and only two prime factors which are separately congruent to unity with respect to one and only one other such factor and the larger of these two factors must also be congruent to unity with respect to the smaller. All such groups are of odd order and the lowest order for which there are exactly three such groups is 29.7.3. When there are exactly three groups of order g and g is divisible by p^2 it cannot obviously be divisible by a higher power of p nor by the square of another prime number. Moreover, no prime factor of g can be congruent to unity with respect to another such factor but one and only one of these primes divides $p+1$. Hence 75 is the only order less than 100 for which there are exactly three groups while there are 25 such orders for which there are exactly two groups.

A necessary and sufficient condition that there are exactly four groups of order g when g is not divisible by the square of a prime number is that g involves either one and only one prime factor which is congruent to unity with respect to another of its prime factors and this is thus congruent to exactly two such factors as was noted above, or that g involves two and only two prime factors each of which is congruent to unity with respect to one and only one other such factor while none of its factors besides the two given ones is congruent to unity with respect to such a factor. It is clear that g cannot be divisi-

ble by the cube of a prime factor nor by the squares of more than two such factors. When it is divisible by the squares of two such factors g must be Abelian and hence a necessary and sufficient condition that there are exactly four groups of such an order g is that the order of a Sylow sub-group diminished by unity cannot be divisible by a prime factor of g.*

When g is divisible by the square of one and only one of its prime factors p then exactly four groups of order g exist when it involves one other factor but not more than one which is congruent to unity with respect to p but not with respect to p^2 or any other prime factor of g while none of its other prime factors nor $p + 2$ is congruent to unity with respect to such a factor. The smallest order for which there are exactly four such groups is 28. It is clear that p cannot be congruent to unity with respect to any one of the prime factors of g and if none of these prime factors is congruent to unity with respect to p then g involves one and only one prime factor which is congruent to unity with respect to one and only one other prime factor. The eight orders less than 100 for each of which there are four and only four groups are therefore 28, 30, 44, 63, 66, 70, 75, and 92.

In 1886, A. B. Kempe published in the London *Philosophical Transactions*, Volume 177, a supposed determination of all the groups whose orders do not exceed 12. This duplicated some of the work of A. Cayley through order 8 but it is not as accurate as the earlier determination since it gives the octic group twice and omits the quaternion group. It also omits the dicyclic group of order 12 and includes permutations which are supposed to generate a group of this order but do not have this property. About three years later, A. Cayley determined correctly all the groups of order 12 in an article published in the *American Journal of Mathematics*, Volume 11, page 144. Hence the first complete determinations of the possible five groups of each of the orders 8 and 12 are due to A. Cayley, whose work relating to the determination of the abstract groups of low orders is much more accurate

* Miller, Blickfeldt and Dickson, *Finite Groups* (1916), p. 117.

than that relating to the determination of the permutation groups of low degrees. This may be partly due to the fact that he was the first to enter the former field but he was not the first in the latter.

The lowest order for which there are more than five groups is 16 and the number of the distinct groups of this order is 14. Notwithstanding the complexity of these groups they were correctly determined in the first publication relating thereto. This publication is due to J. W. A. Young and represents the first American contribution towards the enumeration of all the possible groups of a given order. It appeared in the *American Journal of Mathematics*, Volume 15 (1893), page 124, as a special case relating to groups whose orders are powers of a prime number. Later in the same year, the enumeration of all the possible groups of order 16 appeared also as a special case of an article published by O. Holder in the *Mathematische Annalen*, Volume 43.

While the first two publications relating to the enumeration of the possible groups of order 16 were accurate, the next two such publications were inaccurate. Both of the latter were due to R. Le Vavasseur and appeared in the Paris *Comptes Rendus*, volume 120, page 902; volume 121, page 240, respectively. In the former of these articles the author stated that there are ten and only ten groups of order 16 while in the latter he stated that he had found fifteen such groups. In the following volume of the same periodical he agreed that the corrections made by G. A. Miller to his earlier enumerations of these groups were justified and that the actual number of the groups of order 16 is in fact 14.

A necessary and sufficient condition that two regular permutation groups are simply isomorphic is that they are conjugate and hence every abstract group appears once and only once among the regular permutation groups. That is, the determination of the abstract groups of a given order is equivalent to the determination of the regular permutation groups of that degree. In 1896, G. A. Miller published a list of all of the non-cyclic permutation groups whose degrees are less than 48,

Quarterly Journal of Mathematics, Volume 28, page 254. As there one and only one cyclic group of every order, this list is sufficient to determine all the abstract groups whose orders are less than 48. It is especially noteworthy that the fifteen possible abstract groups of order 24 appear for the first time in this list and that this article contains also the earliest development of the properties of commutator sub-groups, page 266. The groups included in this list about which the widest differences of results were published are those of order 32. Even shortly before the list was published R. Le Vavasseur stated in the Paris *Comptes Rendus*, January 27, 1896, that he had found more than 75 groups of this order and had not yet terminated the enumeration, while their actual number is 51.

About two years after the publication of this list, G. Bagnera considered the general question of determining all the groups of order p^5, p being any prime number, in an article published in the *Annali di Matematica*, 1 (1898), page 137. He concluded therein that there are only 50 groups of order 32 and directed attention to what appeared to him then as an error in the earlier enumeration, but in an article published in the following year in the same periodical he reconsidered this enumeration and agreed that the number of these groups is actually 51, as noted above. The number 32 is the lowest order for which there are more groups than there are units in the order. The number of the possible groups for each of the various higher orders not exceeding 100 were listed by G. A. Miller in the *American Journal of Mathematics*, Volume 52 (1930), page 617, but these groups seem to have not yet been re-determined by others except as they come under general theorems.

G. A. MILLER.

Solutions.

Question 1519.

(R. VAIDYANATHASWAMY) :—ABC is a triangle, I, I_1, I_2, I_3 its in- and ex-centres. If P, P′ are isogonal conjugates, show that (PI, PI_1), (PI_2, PI_3), (PA, PP′) (PB, PC) are pairs of an involution.

Let K, S be the Symmedian point and circum-centre of ABC; SA′, SB′, SC′ perpendiculars to the sides. Prove that (SI, SI_1), (SI_2, SI_3), (SA′, SK), (SB′, SC′) are pairs of an involution.

Solution and Remarks by A. A. Krishnaswami Ayyangar.

The first part is capable of the following generalisation:

PQR is a triangle; A, B, C are points in the sides QR, RP, PQ respectively such that AP, BQ, CR are concurrent at O. T is any point in the plane of the triangle and AU, BV, CW are drawn such that

$$A\,(PR,\ TU) = B\,(QP,\ TV) = C\,(RQ,\ TW) = -1.$$

Then, (i) AU, BV, CW are concurrent;

and (ii) if T′ be the point of concurrence of AU, BV, CW, then (TB, TC), (TA, TT′), (TP, TO), (TQ, TR) are pairs of an involution.

Incidentally, we have to prove that if TP, TQ, TR meet BC, CA, AB at A′, B′, C′ respectively, then AA′, BB′, CC′, TT′ are concurrent: also the relation between T, T′ is involutory, *i.e.*, T, T′ can be interchanged in the above results.

Let us use a system of homogeneous co-ordinates in which ABC is the fundamental triangle and O is (1, 1, 1). Let T be (α, β, γ). Since the equations of AP, AR, AT are respectively

$$y-z=0,\ \ y+z=0,\ \text{ and }\ \gamma y-\beta z=0,$$

and A (PR, TU) $= -1$, the equation of AV is $\beta y - \gamma z = 0$.

Similarly the equation of BV is $\gamma z - \alpha x = 0$,

and of CW is $\alpha x - \beta y = 0$.

Evidently, therefore, AU, BV, CW concur at T $\left(\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}\right)$. The relation between the co-ordinates T, T′* shows the involutory nature of the correspondence. In fact, the points T, T′ are such that the polar lines of either in regard to the conics through the four points P, Q, R, O pass through the other point, and what we have proved here is a well-known result in the geometry of conics (*vide* H. F. Baker's ***Principles of Geometry***, Vol. II, pp. 37, 38, Ex. 15.)

Again, the equation of AA′ is easily seen to be $y(\gamma+\alpha)=z(\alpha+\beta)$ with similar equations for BB′, CC′, which shows that AA′, BB′, CC′ meet at

$$\left(\frac{1}{\beta+\gamma}, \frac{1}{\gamma+\alpha}, \frac{1}{\alpha+\beta}\right).$$

Further, the determinant

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \frac{1}{\alpha} & \frac{1}{\beta} & \frac{1}{\gamma} \\ \frac{1}{\beta+\gamma} & \frac{1}{\gamma+\alpha} & \frac{1}{\alpha+\beta} \end{vmatrix} = \sum \frac{1}{\beta+\gamma}\left(\frac{\beta}{\gamma}-\frac{\gamma}{\beta}\right) = \sum \frac{\beta-\gamma}{\beta\gamma} = 0$$

showing that TT′ also passes through the same point as AA′, BB′, CC′ We shall denote this point by X.

Applying to the quadrilaterals OCPB, AC′XB′ the theorem that the three pairs of lines from any point to the extremities of the three diagonals of any quadrilateral are in involution, we readily perceive that

(TB, TC), (TO, TP), (TQ, TR);

(TB′, TC′), (TA, TX), (TB, TC)

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* It is important to note that T, T′ are isogonal or isotomic conjugates in the triangle ABC according as the homogeneous co-ordinate system used is trilinear or areal, O becoming the in-centre in the first case and the centroid in the second.

are two triplets of involutory pairs. Since TB$'$, TC$'$ are the same lines as TQ, TR, the above triples have two involutory pairs common. Hence the *four* pairs

$$(TB, TC), (TO, TP), (TQ, TR), (TA, TT')$$

(TT$'$ being the same as TX) are in involution.

Particular Cases.

(1) Let O be the in-centre of ABC and P, Q, R the ex-centres. Then since AP is perpendicular to AR, A (PR, TU) = -1 implies that AT, AU are isogonal conjugate lines with respect to triangle ABC. Similar considerations apply to the pairs (BT, BV), and (CT, CW), so that T, T$'$ are isogonal conjugates. Hence, we get the first part of the question if we read P, P$'$, I, I_1, I_2, I_3 in place of T, T$'$, O, P, Q, R respectively in our general theorem.

(2) To prove the second part, let us revert to the notation of the question itself.

Let H be the ortho-centre and G the centroid of the triangle ABC. Let A$'$B$'$C$'$ be the medial triangle of ABC.

It is known that K and S are isotomic conjugates in the triangle A$'$B$'$C$'$ (*vide* Mcclelland's ***Geometry of the Circle***, p 68, Ex. 4).

By a double application of the general theorem, once with S, H as isogonal conjugates in the triangle ABC and a second time with K, S as isotomic conjugates* in the triangle A$'$B$'$C$'$, we get two sets of four pairs in involution, *viz.*,

(i) (SI, SI_1), (SI_2, SI_3), (SB, SC), (SA, SG)

SG being the same line as SH ;

and (ii) (SA, SG), (SB, SC), (SB$'$, SC$'$), (SA$'$, SK).

Since the two sets have two pairs common, the six pairs belong to the same involution.

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* *Vide* foot-note *supra*.

Question 1532.

(V. Ramaswami Aiyar, M.A.) :—Let the tangents at points P, Q of a given conic X intersect in T.

(*a*) Considering the in- and ex- centres of the triangle TPQ, show that if *one* of them lies on the curve, then, ***another*** of them will also lie on the curve.

(*b*) Show that the locus of T in order that such may be the case is a confocal conic Y.

(*c*) Show that the relation between the conics X and Y, in this respect, is reciprocal.

(*d*) If X is a parabola, then Y is the confocal parabola having identical latus-rectum.

Solution by E. H. Neville.

(*a*) If I is the one centre, and J the other centre on TI, and if TI cuts PQ in U, then T and U harmonise with I and J. But T and U harmonise with the two points in which the line TU cuts the conic. Hence if I is one of these points, J is the other.

(*b*) The further developments can be associated with the theory of the Fregier point of a point O on a conic, that is, the point of concurrence of all chords which subtend a right angle at O. Since IPJ and IQJ are right angles, the line TIJ contains the Fregier points of P and Q. Conversely, if the line joining T to the Fregier point of P cuts PQ, the polar of T, in U and cuts the conic in I, J, the lines PI, PJ are perpendicular lines harmonic with respect to PT, PU and are therefore the bisectors of the angles between PT and PU, that is, between PT and PQ. If the same line passes through the Fregier point of Q, then QI, QJ are the bisectors of the angles between QT and QP. Thus,

Two of the contact centres of the triangle TPQ are on the conic if and only if the Fregier points of P and Q are collinear with T.

The solution can now be completed analytically. If F is the Fregier point of P, one chord through F is the diameter which joins

the reflections of P in the two axes, and therefore the mid. point of PF is on the line which joins the projections of P on the two axes.

If the conic is

$$x^2/\alpha + y^2/\beta = 1,$$

the co-ordinates of any point on the normal at P are of the form

$$\{x_p(1-h/\alpha),\ y_p(1-h/\beta)\},$$

and therefore the co-ordinates of the Fregier point F are

$$\{x_p(1-k/\alpha),\ y_p(1-k/\beta)\},$$

when k has such a value that

$$\{x_p(1-\tfrac{1}{2}k/\alpha),\ y_p(1-\tfrac{1}{2}k/\beta)\}$$

is on the line

$$\frac{x}{x_p}+\frac{y}{y_p}=1,$$

that is to say, when

$$\frac{2}{k}=\frac{1}{\alpha}+\frac{1}{\beta}.$$

If (X, Y) is the pole T of PQ, the condition for the two Fregier points to be collinear with T is

$$\begin{vmatrix} X & x_P(1-k/\alpha) & x_Q(1-k/\alpha) \\ Y & y_P(1-k/\beta) & y_Q(1-k/\beta) \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

that is,

$$\begin{vmatrix} X/(\alpha-k) & x_P/\alpha & x_Q/\alpha \\ Y/(\beta-k) & y_P/\beta & y_Q/\beta \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

and since $Xx_P/\alpha + Yy_P/\beta = 1$ and $Xx_Q/\alpha + Yy_Q/\beta = 1$,

this condition is equivalent to

$$X^2/(\alpha-k) + Y^2/(\beta-k) = 1,$$

implying that T is on a fixed conic which manifestly is confocal with the original conic.

(*c*) If, on any scale, OA, OB, OK represent the numbers α, β, k, the relation

$$\frac{2}{k} = \frac{1}{\alpha} + \frac{1}{\beta}$$

represents that O, K are harmonic for A, B. With the origin transferred to K, the same harmonic relation is represented by

$$\frac{2}{-k} = \frac{1}{\alpha - k} + \frac{1}{\beta - k}.$$

Thus to deduce from the locus of T the corresponding locus, we must subtract from $\alpha - k$ and $\beta - k$ the amount $-k$, and so we return to the original conic.

(*d*) On the parabola $y^2 = 4a(x + a)$, the mid. point of PF is the point G in which the normal cuts the axis, and since $NG = 2a$, the co-ordinates of F are $(x_P + 4a, -y_P)$. Thus the condition to be satisfied is

$$\begin{vmatrix} X & x_P + 4a & x_Q + 4a \\ Y & -y_P & -y_Q \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

that is,

$$\begin{vmatrix} X - 2a & x_P + 2a & x_Q + 2a \\ Y & -y_P & -y_Q \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

and since $Yy_P = 2a(X + x_P + 2a)$ and $Yy_Q = 2a(X + x_Q + 2a)$,

this condition is equivalent to

$$Y^2 + 2aX + 2a(X - 2a) = 0,$$

that is, to $$Y^2 = -4a(X - a).$$

Remarks by R. Vaidyanathaswamy.

The proof of (*a*) is immediate, since if X contains an incentre, it must be its own isogonal conjugate with respect to TPQ, and must therefore contain another incentre.

The interesting fact of the symmetry between X, Y belongs to the invariant-theory of two conics.

If S, Ω are two given conics, there is in general a unique conic S' such that Ω is the ϕ-conic of S, S'; if S'' is the polar reciprocal of S' with respect to S, then S, S'', Ω belong to a four-line system, as is immediately evident from the definiton of the ϕ-conic.

If Ω is the Absolute conic (supposed to be non-degenerate) we may obtain a geometrical specification of S''. Let P, Q be points on S, PP_1, PP_2, QQ_1, QQ_2 chords of S which are tangent to Ω. Since Ω is the ϕ-conic of S, S', the polars of P, Q with respect to S' are P_1P_2, Q_1Q_2. Hence if P, Q are such that PQ, P_1P_2, Q_1Q_2 are concurrent, then PQ envelopes S' and T (the meet of the tangents at P and Q) describes S''. As Mr. Neville shews, this is precisely the condition that two of the incentres of TPQ may lie on S.

If Ω is non-degenerate, and the four-line system (S, Ω) is given, the correspondence $S \rightarrow S''$ is easily shewn to be (3, 1), and to have for fixed elements the three point-pairs of the system and the conic Ω. The three conics which correspond to Ω are Ω itself and the two conics of the system outpolar to Ω. We may prove this easily by shewing that if S is outpolar to Ω and Ω' is the polar reciprocal of Ω with respect to S, then Ω is the ϕ-conic of S, Ω'. These conditions do not quite determine the correspondence, but still leave one degree of freedom.

If Ω is a point-pair, then it becomes a repeated singular element of the correspondence, which accordingly becomes (1, 1); it is easily shewn to reduce to the involution determined by the other two point-pairs, which for the case in the question are the two pairs of Loci. For, when Ω is a point-pair, the two conics of the system which are out-polar to Ω, reduce to Ω and its harmonic conjugate with respect to the other two point-pairs, which shews that the (1, 1) correspondence must reduce to the involution determined by the other two point-pairs. This explains the symmetry which arises when Ω reduces to a point-pair.

Question 1536.

(C. N. SRINIVASIENGAR) :—Prove that

(1) The Hessian of a skew ruled cubic surface is the square of a quadric.

(2) The parabolic curve of any skew ruled surface consists of a number of isolated generators.

Solution by the Proposer.

(1) Let $F(x, y, z) = 0$ represent any skew ruled cubic surface. The necessary and sufficient condition that a proper cubic surface should be ruled is that it possesses a double line which is a straight line.* It can be further proved that the surface is skew, for we can prove in various ways that the cubic cone is the only proper cubic surface which is a developable.

Taking the double line as the z-axis, the equation of the cubic may be taken as

$$F \equiv x\phi_1 + y\phi_2 = 0,$$

where $$\phi_r = a_r x^2 + b_r y^2 + c_r z^2 + 2f_r yz + 2g_r zx + 2h_r xy + 2u_r x + 2v_r y + 2w_r z + d_r \ (r = 1, 2)$$

$$\frac{\partial F}{\partial z} \equiv 0 \text{ all along } x = 0,\ y = 0.$$

$$\frac{\partial F}{\partial x} = x\frac{\partial \phi_1}{\partial x} + y\frac{\partial \phi_2}{\partial x} + \phi_1$$

$$\frac{\partial F}{\partial y} = x\frac{\partial \phi_1}{\partial y} + y\frac{\partial \phi_2}{\partial y} + \phi_2.$$

Hence ϕ_1 and ϕ_2 must vanish all along $x = 0,\ y = 0$.

Hence $$c_1 = c_2 = w_1 = w_2 = d_1 = d_2 = 0.$$

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* *Vide* Salmon: *Analytical Geometry of Three Dimensions*, Vol. II, Articles 465 and 520 (Fifth Edition).

The equation of the surface reduces on simplification to the form

$$F = ax^3 + 3bx^2y + 3cxy^2 + dy^3 + z(px^2 + 2qxy + ry^2) + w(lx^2 + 2mxy + ny^2) = 0,$$

where w is a unit variable introduced for the sake of homogeneity.

Evidently $$\frac{\partial^2 F}{\partial z^2} = 0; \quad \frac{\partial^2 F}{\partial z \partial w} = 0; \quad \frac{\partial^2 F}{\partial w^2} = 0.$$

The Hessian is therefore equal to

$$-4 \begin{vmatrix} px + qy & lx + my \\ qx + ry & mx + ny \end{vmatrix}^2$$

(2) The general parametric equations of a ruled surface may be written

$$x = pq + f(p)$$
$$y = \psi(p)q + \chi(p)$$
$$z = q.$$

It is found that $$L = \frac{(q\psi' + \chi')f'' - (q + f')(q\psi'' + \chi'')}{V}$$

$$M = \frac{\chi' - f'\psi'}{V}$$

$$N = 0.$$

Here $V^2 = EG - F^2$; E, F, G and L, M, N represent the fundamental magnitudes of the first and second orders respectively.

The equation $LN - M^2 = 0$ which represents the curves of intersection of the surface with its Hessian reduces to $\chi' - f'\psi' = 0$ or $V =$ infinite. But if V is infinite, we will have $L = 0$, $M = 0$, $N = 0$ which are the conditions for a double line. Hence the parabolic curve is given by $\chi' - f'\psi' = 0$, *i.e.* a function of $p = 0$. The theorem now follows since $p =$ constant represents the generators.

The second part of the question follows from the fact that at any point on a parabolic curve, there can pass only one inflexional tangent and hence only one generator.

[We may construct general examples of ruled surfaces possessing a generator which meets all the other generators. Such a generator which does not belong to the system of generators building up the ruled surface may be called a ***singular generator*** :

If the singular generator also lies on the Hessian, it will be a double line on the ruled surface.

Let us start with a given curve $x = \phi_1(s)$; $y = \phi_2(s)$; $z = \phi_3(s)$.

Then

$$\left.\begin{aligned} x &= \phi_1(s) + \lambda \,.\, r \\ y &= \phi_2(s) + \mu \,.\, r \\ z &= \phi_3(s) + \nu \,.\, r \end{aligned}\right\}$$

may be regarded as the general equations of a ruled surface containing the given curve, where s and r are curvilinear co-ordinates and where λ, μ, ν represent three arbitrary functions of s connected by the relation

$$\lambda^2 + \mu^2 + \nu^2 = 1.$$

Taking ϕ_1, ϕ_2, ϕ_3 as linear functions of s, we get a ruled surface whose generators given by $s =$ constant meet the singular generator given by $r = 0$.

It may be verified that the singular generator does not constitute part of the parabolic curve.

Cubic ruled surfaces furnish a different type of example. Every generator of the cubic must possess one double point, and the locus of double points is a generator which is usually a singular generator.

The surface $xz = y\,\phi\left(\frac{y}{x}\right)$ possesses the singular generator $x = 0$, $y = 0$, which meets all the other generators $y = mx$ $z = m\phi\,(m)$.

If the singular generator lies on the Hessian and meets all the generators, it must be a double line, since there are two inflexional tangents, *viz.*, two generators passing through any point of the generator].

Question 1540.

(S. SIVASANKARANARAYANA PILLAI) :—Let (a, b) denote the greatest common divisor of a and b. Then (i) the necessary and sufficient condition that

$$a^{\phi(k)+1} \equiv a \qquad (\text{mod } k)$$

is that $\left(d, \frac{k}{d}\right) = 1$, where $d = (a, k)$.

(ii) If $(a, k) = d$ and $\left(d, \frac{k}{d}\right) = 1$ and l is the least positive solution of

$$\left.\begin{aligned} x &\equiv 1 \left(\text{mod } \frac{k}{d}\right) \\ x &\equiv 0 \ (\text{mod } d) \end{aligned}\right\}$$

then $a^{\phi(k)} \equiv l \ (\text{mod } k)$.

Solution by P. Jagannathan.

(i) Let $a^{\phi(k)+1} \equiv a \ (\text{mod } k)$ (1)

Then since $(a, k) = d$ $a^{\phi(k)} \equiv 1 \ \left(\text{mod } \frac{k}{d}\right)$

i.e. $$a^{\phi(k)} - n \cdot \frac{k}{d} = 1 \qquad \ldots (2.1)$$

For this, the necessary condition is $\left(a, \frac{k}{d}\right) = 1$

i.e. $$\left(d, \frac{k}{d}\right) = 1.$$

For, if $\left(a, \frac{k}{d}\right) = g > 1$, one side of (2.1) is divisible by g and the other not.

Next, if $\left(d, \frac{k}{d}\right) = 1$, then $\left(a, \frac{k}{d}\right) = 1$ and by Fermat's theorem

$$a^{\phi(k/d)} \equiv 1 \ \left(\text{mod. } \frac{k}{d}\right)$$

Raising both sides to the power $\phi(d)$

$$a^{\phi(k)} \equiv 1 \left(\bmod.\ \frac{k}{d}\right)$$

i.e., $$a^{\phi(k)+1} \equiv a \ (\bmod.\ k)$$

Hence the condition is sufficient.

(ii) 'l' is the least positive solution of the congruences

$$x \equiv 1 \left(\bmod.\ \frac{k}{d}\right) \quad \ldots \quad (3)$$

and $$x \equiv 0 \ (\bmod.\ d). \quad \ldots \quad (4)$$

From (3) $$l \equiv 1 \left(\bmod.\ \frac{k}{d}\right)$$

but $$a^{\phi(k)} \equiv 1 \left(\bmod.\ \frac{k}{d}\right) \text{ from (2)}$$

$\therefore$ $$a^{\phi(k)} \equiv l \left(\bmod.\ \frac{k}{d}\right) \quad \ldots \quad (5)$$

From (4) $$l \equiv 0 \ (\bmod.\ d).$$

but evidently $$a^{\phi(k)} \equiv 0 \ (\bmod.\ d)$$

$\therefore$ $$a^{\phi(k)} \equiv l \ (\bmod.\ d). \quad \ldots \quad (6)$$

From (5) and (6)

since $$\left(d, \frac{k}{d}\right) = 1,$$

$$a^{\phi(k)} \equiv l \ (\bmod.\ k).$$

Questions for Solution.

Proposers of Questions are requested to send their own solutions along with their questions.

1616. (K. J. SANJANA, M.A.) :—A variable chord of the ellipse

$$E \equiv a^2b^2 - b^2x^2 - a^2y^2 = 0$$

is of given length c less than $2b$ and is divided into two parts lc, mc, in the constant ratio $l:m$ where $l + m = 1$.

(1) Prove that the locus of the point of division is

$$a^2y\,(lmE - l^2m^2b^2c^2)^{\frac{1}{2}} + b^2x\,(l^2m^2a^2c^2 - lmE)^{\frac{1}{2}} = \tfrac{1}{2}\,(l-m)\,(a^2 - b^2)^{\frac{1}{2}}E.$$

(2) Deduce the equation of the locus of the middle point of the chord.

(3) Prove further that in both cases the area enclosed by the locus is $\pi\ (ab - lmc^2)$.

1617. (R. VAIDYANATHASWAMY) :—The θ-normal at a point P on a conic is defined to be the line obtained by rotating the tangent at P through an angle θ in the positive direction about P. A θ-normal which passes through the centre of the conic is a θ-diameter.

If the sides of a triangle inscribed in a conic are respectively parallel to θ_1 -, θ_2 -, and θ_3 -diameters of the conic, shew that the θ-normals to the conic at its vertices are concurrent, θ being given by

$$\cot\theta = \cot\theta_1 + \cot\theta_2 + \cot\theta_3 .$$

1618. (R. VAIDYANATHASWAMY) :—The intersection of consecutive θ-normals at P is called the centre of θ-curvature at P. Prove that the θ-normal at a point P on a conic contains the centres of θ-curvature at four other points $Q_1Q_2Q_3Q_4$. Prove further that the $(-\theta)$-normals at $Q_1Q_2Q_3Q_4$ are concurrent.

1619. (P. GANAPATI):—The locus of the mid. points of diameters (the lines joining pairs of points where the tangents are parallel) of an oval with finite continuous curvature has at least three cusps.

1620. (K. J. SANJANA, M.A.):—Find the form of the function $f(x)$ which satisfies the differential equation

$$\{f(x)\}^2 - 2\cos A \,.\, f(x) + 1 = af'(x),$$

where A and a are constants.

1621. (K. J. SANJANA, M.A.):—(*a*) Sum the infinite series

$$1 \pm \frac{n^2}{2.5} + \frac{n^4}{2.4.5.7} \pm \frac{n^6}{2.4.6.5.7.9} + \frac{n^8}{2.4.6.8.5.7.9.11} + \dots\dots$$

(*b*) Show how to find the sum of the more general series

$$1 + \frac{n^2}{k.l} + \frac{n^4}{k(k+a).l(l+a)} + \frac{n^6}{k(k+a)(k+2a).l(l+a)(l+2a)} + \dots\dots,$$

where k, l, $l-k$, a are all positive integers.

1622. (A. RANGANATHA RAO):—The normals at four points ABCD on a conic S are concurrent at O:

(i) Prove that the circumcentres of ABC, BCD, CDA, DAB, lie on a rectangular hyperbola whose asymptotes are the principal axes of S.

(ii) If D is a fixed point and O a variable point on the normal at D, the circle ABC generates a co-axial system whose radical axis is a tangent to S.

1623. (HANSRAJ GUPTA):—Prove that

$$\frac{1!\,2!\,3!\dots(r-1)!\,(nr)!}{n!\,(n+1)!\,(n+2)!\dots(n+r-1)!}$$

is an integer for all positive integral values of n and r.

LIST OF JOURNALS & BOOKS RECEIVED IN THE LIBRARY

during the months of April and May 1932.

1 Acta Mathematica, **58**, 3, 4.
2 American Journal of Mathematics, **54**, 2 (3 copies).
3 American Mathematical Monthly, **39**, 1 (2 copies), **39**, 3 (3 copies), **39**, 4 (3copies).
4 Annales de l'Ecole Normale Superieure, 1, 2, 3 (1932).
5 Annals of Mathematics, **33**, 2 (2 copies).
6 Astrophysical Journal, **75**, 1, 2.
7 Bulletin of the American Mathematical Society, **38**, 3, 4.
8 Bulletin des Sciences Mathematiques, **56**, Mar. & April 1932.
9 Crelle's Journal, **166**, 3.
10 Jahresbericht der deutsche mathematiker Vereinigung, **41**, 5, 6, 7, 8.
11 Japanese Journal of Mathematics, **8**, 4.
12 Mathematical Gazette, **16**, 218.
13 Mysore Half-Yearly Journal, **5**, 1.
14 Mathematische Annalen, **106**, 2, 3.
15 Monthly Notices ot the Royal Astronomical Society, **92**, 4, 5.
16 Philosophical Magazine, **13**, 85, 86, 87.
17 Philosophical Transactions of the Royal Society, **231**, pp. 1—27.
18 Popular Astronomy, **39**, 1—10, **40**, 3, 4.
19 Proceedings of the Edinburgh Mathematical Society, **3**, 1 (3 copies).
20 Proceedings of the London Mathematical Society, **33**, 5, **33**, 6.
21 Proceedings of the Physico-Mathematical Society of Japan, **13**, 2, 9, **14**, 2.
22 Proceedings of the Royal Society of London, **135**, 827, 828.
23 Quarterly Journal of Mathematics, **3**, 9.
24 Revue Semestrielle des Publications Mathmatiques, **36**.

Books.

1 Calendar of the University of Madras, 1931-1932, Vol. 2.
2 Sur la Solution approchee des Problemes de la Physique Mathematique et de la Science d'Ingenieur.
3 Progres recents dans l'Integration approchee des Equations de la Physique Mathematique.
4 Application de la Methode de l'algorithme variationnel a' la solution approchee des equations differentielles aux derivees partielles du type elliptique.
5 Sur Quelques Recherches Recentes dans le Domaine de la Solution Approchee des Problemes de la Physique Mathematique.
6 Sur Quelques Idees de P. Tchebycheff qui peuvent etre rattachee's a la Solution approchee des Problemes du Calcul des Variations.
7 Les Problemes Fondamentaux de la Physique Mathematique et de la Science d'Ingineur.

Numbers in black type refer to the volumes, and those in ordinary type to the numbers of the issues.

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